

TSIRELSON LIKE OPERATOR SPACES

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ABSTRACT. We construct nontrivial examples of weak- C_p ($1 \leq p \leq \infty$) operator spaces with the local operator space structure very close to $C_p = [R, C]_{\frac{1}{p}}$. These examples are non-homogeneous Hilbertian operator spaces, and their constructions are similar to that of 2-convexified Tsirelson's space by W. B. Johnson.

1. INTRODUCTION

Tsirelson's space and its variations have been sources of counterexamples to many questions in the Banach space theory (See [1] for the details). In this paper we focus on the 2-convexified version of Tsirelson's space, which served as an example of nontrivial weak Hilbert space, a closest object to Hilbert spaces in the sense of type and cotype theory.

Recall that a Banach space X is called a weak Hilbert space ([6]) if for any $0 < \delta < 1$ there is a constant $C > 0$ with the following property : for any finite dimensional $F \subseteq X$ we can find $F_1 \subseteq F$ and an onto projection $P : X \rightarrow F_1$ satisfying

$$d_{F_1} := d(F_1, \ell_2^{\dim F_1}) \leq C, \dim F_1 \geq \delta \dim F \text{ and } \|P\| \leq C,$$

where $d(\cdot, \cdot)$ is the Banach-Mazur distance defined by

$$d(X, Y) = \inf\{\|u\| \|u^{-1}\| \mid u : X \rightarrow Y, \text{ isomorphism}\}.$$

As an operator space analogue of weak Hilbert space the author introduced the notion of weak- H spaces for a (separable and infinite dimensional) perfectly Hilbertian operator space H in [5]. A Hilbertian operator space H (i.e. H is isometric to a Hilbert space) is called homogeneous if for every $u : H \rightarrow H$ we have $\|u\|_{cb} = \|u\|$ and subquadratic if for all orthogonal projections $\{P_i\}_{i=1}^n$ in H with $I_H = P_1 + \cdots + P_n$ we have

$$\|x\|_{B(\ell_2) \otimes_{\min} H}^2 \leq \sum_{i=1}^n \|I_{B(\ell_2)} \otimes P_i(x)\|_{B(\ell_2) \otimes_{\min} H}^2$$

for any $x \in B(\ell_2) \otimes H$ (See p.82 of [9]), where \otimes_{\min} is the injective tensor product of operator spaces. A homogeneous Hilbertian operator space H is called perfectly Hilbertian if H and H^* is subquadratic. See [4] for the definition of weak- H space and the related type and cotype notions of operator spaces.

In [5] it is shown that an operator space E is a weak- H space if and only if for any $0 < \delta < 1$ there is a constant $C > 0$ with the following property : for any finite dimensional $F \subseteq E$ we can find $F_1 \subseteq F$ and an onto projection $P : E \rightarrow F_1$ satisfying

$$d_{F_1, cb}^H := d_{cb}(F_1, H_{\dim F_1}) \leq C, \dim F_1 \geq \delta \dim F \text{ and } \|P\|_{cb} \leq C,$$

where $d_{cb}(\cdot, \cdot)$ is the cb-distance defined by

$$d_{cb}(E, F) = \inf\{\|u\|_{cb} \|u^{-1}\|_{cb} \mid u : E \rightarrow F, \text{ isomorphism}\}$$

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and H_n is the n -dimensional subspace of H . Thus, we can say that weak- H spaces have similar local operator space structure to H .

The aim of this paper is to construct nontrivial examples of weak- H spaces (not completely isomorphic to H) for $H = C_p, R_p$ and $1 \leq p \leq \infty$, where $C_p = [C, R]_{\frac{1}{p}}$ and $R_p = [R, C]_{\frac{1}{p}}$, interpolation spaces of the column and the row Hilbert space via complex method. We will follow the approach of W. B. Johnson (and T. Figiel) to construct Hilbertian operator spaces X_{C_p} and X_{R_p} for $1 \leq p \leq \infty$. Since $R_p^* = C_{p'}$ for $\frac{1}{p} + \frac{1}{p'} = 1$ it is enough to consider X_{C_p} and X_{R_p} for $1 \leq p \leq 2$. Interestingly, the construction for the case $1 \leq p < 2$ and the case $p = 2$ are different although many proofs are overlapping.

In section 2 we prepare some back ground materials concerning vector valued Schatten classes and a description of $d_{E,cb}^H$. In section 3 we focus on $1 \leq p < 2$ case. In section 3.1 we will construct X_{C_p} (resp. X_{R_p}) and investigate the behavior of its canonical basis. It will be shown that the span of certain block sequences of the canonical basis is completely isomorphic to C_p (resp. R_p) of the same dimension with bounded constants. In section 3.2 we will examine that X_{C_p} (resp. X_{R_p}) is our desired space. First, we show that X_{C_p} (resp. X_{R_p}) is a weak- C_p (resp. weak- R_p). We prepare additional materials concerning $\pi_{2,H}$ -norms, operator space analogue of absolutely 2-summing norm, and related description of $d_{E,cb}^H$. Secondly, it will be shown that X_{C_p} (resp. X_{R_p}) is not completely isomorphic to C_p (resp. R_p) investigating containment of an isomorphic copy of c_0 (the Banach space of all sequences vanishing at infinity) as in chapter 13 of [7].

In section 4 we consider $p = 2$ case. Most of arguments from section 3 are still available with some exceptions. In the final remark we construct a non-Hilbertian example of weak- OH space.

Throughout this paper, we assume that the reader is familiar with basic concepts in Banach spaces ([2, 7, 11]) and operator spaces ([3, 10]). In this paper E and H will be reserved for an operator space and a separable, infinite dimensional and perfectly Hilbertian operator space. Note that $H(I)$ is well-defined for any index set I ([8]). We will simply write H_n when $I = \{1, \dots, n\}$. As usual, $B(E, F)$ and $CB(E, F)$ denote the set of all bounded linear maps and all cb-maps from E into F , respectively.

\mathcal{K} implies the algebra of compact operators on ℓ_2 and \mathcal{K}_0 is the union of the increasing sequence $M_1 \subseteq \dots \subseteq M_n \subseteq M_{n+1} \subseteq \dots$ of matrix algebras. Note that $\mathcal{K} = \overline{\mathcal{K}_0}$.

2. PRELIMINARIES

In this section we collect some back ground materials which will be used later.

2.1. Vector valued Schatten classes. Let $S_p(\mathcal{H})$ be the Schatten class on a Hilbert space \mathcal{H} and $1 \leq p \leq \infty$. In [9] $S_p(\mathcal{H}, E)$, E -valued Schatten classes are defined by

$$S_p(\mathcal{H}; E) := [S_\infty(\mathcal{H}) \otimes_{\min} E, S_1(\mathcal{H}) \widehat{\otimes} E]_{\frac{1}{p}},$$

where \otimes_{\min} and $\widehat{\otimes}$ refer to injective and projective tensor products of operator spaces. When $\mathcal{H} = \ell_2$ we simply write as S_p and $S_p(E)$.

The above vector valued Schatten classes are useful to describe operator space structure of subspaces of S_p . In particular, the operator space structure of C_p, R_p and ℓ_p (c_0 when $p = \infty$) can be described as follows.

$$\left\| \sum_i x_i \otimes e_i \right\|_{S_p(C_p)} = \left\| \left(\sum_i x_i^* x_i \right)^{\frac{1}{2}} \right\|_{S_p},$$

$$\left\| \sum_i x_i \otimes e_i \right\|_{S_p(R_p)} = \left\| \left(\sum_i x_i x_i^* \right)^{\frac{1}{2}} \right\|_{S_p}$$

and

$$\left\| \sum_i x_i \otimes e_i \right\|_{S_p(\ell_p)} = \left(\sum_i \|x_i\|_{S_p}^p \right)^{\frac{1}{p}}.$$

When we are dealing with $E +_p F$, the sum of operator spaces in the sense of ℓ_p , it is more appropriate to use vector valued Schatten classes. See chapter 2 of [9] for the definition of $E \oplus_p F$, the direct sum of operator spaces in the sense of ℓ_p . Then we define $E +_p F := (E \oplus_p F)/(E \cap F)$. Then we have

$$(2.1) \quad S_p(E \oplus_p F) = S_p(E) \oplus_p S_p(F) \text{ and } S_p(E +_p F) = S_p(E) +_p S_p(F).$$

Moreover, we can use vector valued Schatten classes to check complete boundedness. Indeed, by Lemma 1.7. of [9] for any cb-map $T : E \rightarrow F$ between operator spaces we have

$$(2.2) \quad \|T\|_{cb} = \sup_{n \geq 1} \left\| I_{S_p^n} \otimes T : S_p^n(E) \rightarrow S_p^n(F) \right\|.$$

For a subspace $S \subseteq S_p$ we denote $S(E) := \overline{\text{span}}\{S \otimes E\} \subseteq S_p(E)$. We will focus on the case $S = C_p$, R_p and ℓ_p (c_0 when $p = \infty$). By Theorem 1.1. of [9] we have

$$S_p(E) \cong C_p \otimes_h E \otimes_h R_p$$

completely isometrically by the identification $e_{ij} \otimes x \mapsto e_{i1} \otimes x \otimes e_{1j}$, so that $C_p(E)$ and $R_p(E)$ are 1-completely complemented in $S_p(E)$.

By taking the natural diagonal projection $\ell_p(E)$ is also 1-completely complemented in $S_p(E)$, and we have

$$\ell_p(E) = [c_0 \otimes_{\min} E, \ell_1 \widehat{\otimes} E]_{\frac{1}{p}}.$$

By the above observations for any cb-map $T : E \rightarrow F$ between operator spaces we have

$$(2.3) \quad \|I_S \otimes T : S(E) \rightarrow S(F)\|_{cb} = \|T\|_{cb}$$

for $S = C_p$, R_p and ℓ_p (c_0 when $p = \infty$).

2.2. A description of $d_{E,cb}^H$ and operator spaces with similar n -dimensional structure to H . In this section we present several observation concerning $(2, H)$ -summing maps, operator space analogues of 2-summing maps, and related description of $d_{E,cb}^H$.

For a linear map $T : E \rightarrow F$ between operator maps the $(2, H)$ -summing norm $\pi_{2,H}(T)$ is defined by

$$\pi_{2,H}(T) := \sup \left\{ \frac{(\sum_k \|T S e_k\|^2)^{\frac{1}{2}}}{\|S : H^* \rightarrow E\|_{cb}} \right\}.$$

We need the subquadracity of H to ensure that $\pi_{2,H}(\cdot)$ is a norm.

The factorization norm through H of T , $\gamma_H(T)$, is defined by

$$\gamma_H(T) := \inf \{ \|T_1\|_{cb} \|T_2\|_{cb} \},$$

where the infimum runs over all possible factorization

$$T : E \xrightarrow{T_1} H^*(I) \xrightarrow{T_2} F$$

for some index set I . Note that the trace dual γ_H^* of γ_H can be described as follows (Theorem 6.1. of [8]). For any finite rank map $T : E \rightarrow F$ we have

$$\gamma_H^*(T) = \inf \{ \pi_{2,H^*}(T_1) \pi_{2,H}(T_2^*) \},$$

where the infimum runs over all possible factorization

$$T : E \xrightarrow{T_1} \ell_2^m \xrightarrow{T_2} F$$

for some $m \in \mathbb{N}$.

We need to consider completely nuclear norm $\nu^o(\cdot)$ ([3]), which is the trace dual of cb-norm $\|\cdot\|_{cb}$. By arguments in p.200-201 of [3] we have

$$(2.4) \quad |\text{tr}(T)| \leq \nu^o(T)$$

for all finite rank map $T : E \rightarrow E$.

The following Lemma is an operator space version of the fact that the composition of two 2-summing maps is a nuclear map in some special cases.

Lemma 2.1. *Let H be a subquadratic, homogeneous and Hilbertian operator space and E be a finite dimensional operator space. Then for any $u : E \rightarrow E$ we have*

$$\nu^o(u) \leq \inf\{\pi_{2,H}(S)\pi_{2,H}^*(T)\},$$

where the infimum runs over all possible factorization

$$u : E \xrightarrow{S} \ell_2^n \xrightarrow{T} E.$$

Proof. Let $u : E \rightarrow E$ and consider any factorization $u : E \xrightarrow{S} \ell_2^n \xrightarrow{T} E$. Then for any $v : E \rightarrow E$ and any further factorization $T : \ell_2^n \xrightarrow{A} H^* \xrightarrow{B} E$ we have

$$\begin{aligned} |\text{tr}(vu)| &= |\text{tr}(vBAS)| = |\text{tr}(SvBA)| \leq \|A\|_{HS} \|SvB\|_{HS} \\ &= \|A\|_{HS} \pi_{2,H}(SvB) \leq \|A\|_{HS} \|B\|_{cb} \|v\|_{cb} \pi_{2,H}(S). \end{aligned}$$

By taking infimum over all possible A, B, S and v with $\|v\|_{cb} \leq 1$ we get the desired result using trace duality. \square

Now we present a description of $d_{E,cb}^H$ using $(2, H)$ -summing norms.

Lemma 2.2. *Let H be a perfectly Hilbertian operator space. Then we have*

$$d_{E,cb}^H = \sup \left\{ \frac{\pi_{2,H^*}^*(u)}{\pi_{2,H}(u^*)} \mid u : \ell_2^m \rightarrow E, m \in \mathbb{N} \right\}.$$

Proof. We get the result from Theorem 4.3. of [4] and the fact that

$$\pi_{2,H}(u^*) \leq \ell(u) \leq \pi_{2,H^*}^*(u)$$

for any $u : \ell_2^m \rightarrow E$. \square

We show an operator space version of Remark 13.4. of [8], which will be useful later.

Proposition 2.3. *Let H be a perfectly Hilbertian operator space and $n \in \mathbb{N}$ be fixed. Suppose that E is an operator space satisfying the following : there is a constant $C > 0$ such that for any n -dimensional subspace $F \subseteq E$ we have*

$$d_{F,cb}^H \leq C.$$

Then for any n -dimensional subspace $F \subseteq E$ we have a projection $P : E \rightarrow E$ onto F with

$$\gamma_H(P) \leq C.$$

Proof. By combining Lemma 2.2 and the assumption we get

$$(2.5) \quad \pi_{2,H^*}^*(T) \leq C \cdot \pi_{2,H}(T^*)$$

for any $T : \ell_2^n \rightarrow E$.

Now we fix a n -dimensional subspace $F \subseteq E$ and let $i : F \hookrightarrow E$ be the inclusion. For any $u : F \rightarrow F$ we consider a factorization

$$iu : F \xrightarrow{\alpha} \ell_2^m \xrightarrow{\beta} E.$$

Then by applying Lemma 2.1, (2.4) and (2.5) we have

$$\begin{aligned} |\mathrm{tr}(u)| &\leq \nu^o(u) \leq \pi_{2,H^*}(\alpha) \pi_{2,H^*}^*(\beta|_{\mathrm{ran}\alpha}) \\ &\leq C \cdot \pi_{2,H^*}(\alpha) \pi_{2,H}((\beta|_{\mathrm{ran}\alpha})^*) \\ &\leq C \cdot \pi_{2,H^*}(\alpha) \pi_{2,H}(\beta^*). \end{aligned}$$

By taking infimum over all possible α and β we get

$$|\mathrm{tr}(u)| \leq C \cdot \gamma_H^*(iu).$$

If we apply Hahn-Banach theorem to the functional $u \mapsto \mathrm{tr}(u)$ we get the desired result. \square

3. THE CASE $1 \leq p < 2$

3.1. The construction and basic properties of the canonical basis. In this section we will construct Hilbertian operator spaces X_{C_p} and X_{R_p} for $1 \leq p < 2$, consequently X_{C_p} and X_{R_p} for $1 \leq p \leq \infty, p \neq 2$. We will mainly focus on X_{C_p} case only since the situation of X_{R_p} is symmetric.

We say that a disjoint collection, $(E_j)_{j=1}^{f(k)}$, of finite subsets of \mathbb{N} is “allowable” if

$$E_j \subseteq \{k, k+1, \dots\} \text{ for all } 1 \leq j \leq f(k),$$

where $k \in \mathbb{N}$ and $f(k) = (4k^3)^k$. This specific choice of f will be clarified later in section 4.2. For a finite subset $E \subseteq \mathbb{N}$ and

$$x = \sum_{i \geq 1} x_i \otimes t_i \in \mathcal{K}_0 \otimes c_{00},$$

where t_i is the i -th unit vector in c_{00} (finitely supported sequences of complex numbers), we denote

$$Ex = \sum_{i \in E} x_i \otimes t_i.$$

Let $1 \leq p < 2$ and $0 < \theta < 1$ be fixed. We will define a sequence of norms on $\mathcal{K}_0 \otimes c_{00}$ to construct X_{C_p} (resp. X_{R_p}). For $x \in \mathcal{K}_0 \otimes c_{00}$ we define

$$\|x\|_0 = \|x\|_{\mathcal{K} \otimes_{\min}(R_p +_p C_p)}.$$

Then X_0 , the completion of $(c_{00}, \|\cdot\|_0)$, is nothing but the homogeneous Hilbertian operator space $R_p +_p C_p$, and clearly $\|\cdot\|_0$ satisfies Ruan’s axioms. Now we define $(\|\cdot\|_n)_{n \geq 0}$ recursively. Suppose that $\|\cdot\|_n$ is already defined and satisfies Ruan’s axioms ([3, 10]). Then X_n , the completion of $(c_{00}, \|\cdot\|_n)$ is an operator space, and we define

$$\begin{aligned} \|x\|_{n+1} &= \max \left\{ \|x\|_n, \theta \sup \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x \right\|_{\mathcal{K} \otimes_{\min} C_p(X_n)} \right\} \\ &\quad (\text{resp. } \max \left\{ \|x\|_n, \theta \sup \left\| \sum_{j=1}^{f(k)} e_{1j} \otimes E_j x \right\|_{\mathcal{K} \otimes_{\min} R_p(X_n)} \right\}), \end{aligned}$$

where the inner supremum runs over all “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$. Then $\|\cdot\|_{n+1}$ satisfies Ruan’s axioms, so that X_{n+1} , the completion of $(c_{00}, \|\cdot\|_{n+1})$ is an operator space. Actually, X_{n+1} is a subspace of $X_n \oplus_\infty \ell_\infty(I; \{C_p(X_n)\})$ spanned by elements of the form, $(x, (\theta E_j x)_{(E_j) \in I})$, where I is the collection of all allowable sequences, so that X_{n+1} inherits the operator space structure from $X_n \oplus_\infty \ell_\infty(I; \{C_p(X_n)\})$ (the case for R_p is similar).

Remark 3.1. When we write $e_{j1} \otimes E_j x \in \mathcal{K} \otimes_{\min} C_p(X_n)$ one should note that $e_{j1} \in C_p$ and $E_j x \in \mathcal{K} \otimes X_n$, which is twisted in order.

Proposition 3.2. *For any $x \in \mathcal{K}_0 \otimes c_{00}$, $(\|x\|_n)_{n \geq 0}$ is increasing, and we have*

$$\|x\|_{\mathcal{K} \otimes_{\min}(R_p + {}_p C_p)} \leq \|x\|_n \leq \|x\|_{\mathcal{K} \otimes_{\min} C_p}$$

for all $n \geq 0$.

Proof. The left inequality is clear. For the right inequality we use induction on n . When $n = 0$, it is trivial. Suppose we have the right inequality for n and for all $x \in \mathcal{K}_0 \otimes c_{00}$, equivalently, the formal identity $C_p \rightarrow X_n$ is completely contractive. Then we have

$$\begin{aligned} \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x \right\|_{S_p(C_p(X_n))} &= \theta \left\| \sum_{j=1}^{f(k)} \sum_{i \in E_j} x_i \otimes e_i \otimes e_{j1} \right\|_{S_p(C_p(X_n))} \\ &\leq \theta \left\| \sum_{j=1}^{f(k)} \sum_{i \in E_j} x_i \otimes e_{i1} \otimes e_{j1} \right\|_{S_p(C_p(C_p))} \\ &= \theta \left\| \left(\sum_{j=1}^{f(k)} \sum_{i \in E_j} x_i^* x_i \right)^{\frac{1}{2}} \right\| \\ &\leq \theta \|x\|_{S_p(C_p)} < \|x\|_{S_p(C_p)}. \end{aligned}$$

Thus, we have that

$$x \mapsto \theta \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x, \quad C_p \rightarrow C_p(X_n)$$

is a complete contraction, and so is

$$x \mapsto x \oplus \theta \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x, \quad C_p \rightarrow X_n \oplus_{\infty} C_p(X_n)$$

by the assumption, which leads us to the right inequality for $n + 1$ and for all $x \in \mathcal{K}_0 \otimes c_{00}$. \square

Now we can consider $\|x\| = \lim_{n \rightarrow \infty} \|x\|_n$ for all $x \in \mathcal{K}_0 \otimes c_{00}$, and clearly $\|\cdot\|$ satisfies Ruan's axioms, so that X_{C_p} (resp. X_{R_p}), the completion of $(c_{00}, \|\cdot\|)$ is an operator space. Actually, X_{C_p} (resp. X_{R_p}) is a subspace of $\ell_{\infty}(X_n)$ spanned by elements of the form, (x, x, \dots, x) , so that X_{C_p} (resp. X_{R_p}) inherits the operator space structure from $\ell_{\infty}(X_n)$. Moreover, X_{C_p} (resp. X_{R_p}) is Hilbertian by Proposition 3.2.

We have a slight different form of $\|\cdot\|_n$ which will be useful later.

Proposition 3.3. *For any $x \in \mathcal{K}_0 \otimes c_{00}$ and any $n \geq 0$ we have*

$$\|x\|_{n+1} = \max \left\{ \|x\|_0, \theta \sup \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x \right\|_{\mathcal{K} \otimes_{\min} C_p(X_n)} \right\},$$

where the inner supremum runs over all “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$.

Proof. Suppose we have

$$\|x\|_{n+1} > \theta \sup \left\{ \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x \right\|_{\mathcal{K} \otimes_{\min} C_p(X_n)} \right\}$$

for an $x \in \mathcal{K}_0 \otimes c_{00}$. Then by the definition of $\|\cdot\|_{n+1}$ we have $\|x\|_{n+1} = \|x\|_n$. Since the formal identity $i_n : X_n \rightarrow X_{n-1}$ is clearly a complete contraction we get another complete contraction

$$I_{C_p} \otimes i_n : C_p(X_n) \rightarrow C_p(X_{n-1})$$

by (2.3). Thus, it follows that

$$\begin{aligned} \|x\|_n &> \theta \sup \left\{ \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x \right\|_{\mathcal{K} \otimes \min C_p(X_n)} \right\} \\ &\geq \theta \sup \left\{ \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x \right\|_{\mathcal{K} \otimes \min C_p(X_{n-1})} \right\} \end{aligned}$$

and hence $\|x\|_n = \|x\|_{n-1}$. If we repeat this process, then we get $\|x\|_{n+1} = \|x\|_0$. \square

We say that a basis $\{f_i\}_{i \geq 1}$ of an operator space E is C -completely unconditional if

$$\left\| \sum_{i \geq 1} a_i x_i \otimes f_i \right\|_{\mathcal{K} \otimes \min E} \leq C \left\| \sum_{i \geq 1} x_i \otimes f_i \right\|_{\mathcal{K} \otimes \min E}$$

for any finitely supported sequence of $\{x_i\}_{i \geq 1} \subseteq \mathcal{K}_0$ and any sequence of scalars $(a_i)_{i \geq 1}$ with $|a_i| \leq 1$ for all $i \geq 1$.

Proposition 3.4. *The canonical basis $\{t_i\}_{i \geq 1}$ is a normalized 1-completely unconditional basis for X_{C_p} .*

Proof. We will use induction on n , to show that $\{t_i\}_{i \geq 1}$ is a normalized 1-completely unconditional basis for X_n . First, we fix a sequence $(a_i)_{i \geq 1}$ with $|a_i| \leq 1$ for all $i \geq 1$.

When $n = 0$, for any $\sum_{i \geq 1} x_i \otimes e_i \in \mathcal{K}_0 \otimes c_{00}$ we have

$$\begin{aligned} &\left\| \sum_{i \geq 1} a_i x_i \otimes t_i \right\|_{S_p(R_p + {}_p C_p)}^p \\ &= \inf_{x_i = y_i + z_i} \left\{ \left\| \left(\sum_{i \geq 1} |a_i|^2 y_i y_i^* \right)^{\frac{1}{2}} \right\|_{S_p}^p + \left\| \left(\sum_{i \geq 1} |a_i|^2 z_i^* z_i \right)^{\frac{1}{2}} \right\|_{S_p}^p \right\} \\ &\leq \inf_{x_i = y_i + z_i} \left\{ \left\| \left(\sum_{i \geq 1} y_i y_i^* \right)^{\frac{1}{2}} \right\|_{S_p}^p + \left\| \left(\sum_{i \geq 1} z_i^* z_i \right)^{\frac{1}{2}} \right\|_{S_p}^p \right\} \\ &= \left\| \sum_{i \geq 1} x_i \otimes t_i \right\|_{S_p(R_p + {}_p C_p)}^p, \end{aligned}$$

since $|a_i|^2 y_i y_i^* \leq y_i y_i^*$ and $|a_i|^2 z_i^* z_i \leq z_i^* z_i$.

Suppose we have the result for n , which is equivalent to

$$X_n \rightarrow X_n, t_i \mapsto a_i t_i$$

is a complete contraction. Then by (2.3) we have another complete contraction

$$C_p(X_n) \rightarrow C_p(X_n), e_{j1} \otimes t_i \mapsto e_{j1} \otimes a_i t_i.$$

Thus, for any “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$ we have

$$\begin{aligned}
& \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j \left(\sum_i a_i x_i \otimes t_i \right) \right\|_{\mathcal{K} \otimes \min(C_p(X_n))} \\
&= \theta \left\| \sum_{j=1}^{f(k)} \sum_{i \in E_j} x_i \otimes e_{j1} \otimes a_i t_i \right\|_{\mathcal{K} \otimes \min(C_p(X_n))} \\
&\leq \theta \left\| \sum_{j=1}^{f(k)} \sum_{i \in E_j} x_i \otimes e_{j1} \otimes t_i \right\|_{\mathcal{K} \otimes \min(C_p(X_n))} \\
&= \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j \left(\sum_i x_i \otimes t_i \right) \right\|_{\mathcal{K} \otimes \min(C_p(X_n))} \leq \left\| \sum_i x_i \otimes t_i \right\|_{n+1},
\end{aligned}$$

which implies the result for $n+1$. \square

Now we investigate the operator space structure of the subspace spanned by certain normalized and disjoint block sequences of $\{t_i\}_{i \geq 1} \subseteq X_{C_p}$. They are θ -completely isomorphic to C_p with the same dimension.

Proposition 3.5. *Let $(y_j)_{j=1}^{f(k)}$ be a disjoint and normalized block sequences of $\{t_i\}_{i \geq 1} \subseteq X_{C_p}$ with $\text{supp } y_j \subseteq \{k, k+1, \dots\}$ for $1 \leq j \leq f(k)$. Then we have*

$$\theta \left\| \sum_{j=1}^{f(k)} b_j \otimes e_{j1} \right\|_{S_p(C_p)} \leq \left\| \sum_{j=1}^{f(k)} b_j \otimes y_j \right\|_{S_p(X_{C_p})} \leq \left\| \sum_{j=1}^{f(k)} b_j \otimes e_{j1} \right\|_{S_p(C_p)}$$

for any $(b_j)_{j=1}^{f(k)} \subseteq S_p$.

Proof. For the left inequality we set $E_j = \text{supp } y_j$ and $n_j = \min\{\text{supp } y_j\}$. Since

$$X_{C_p} \rightarrow C_p(X_n), \quad x \mapsto \theta \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x$$

is a complete contraction we have

$$\begin{aligned}
\left\| \sum_{i=1}^{f(k)} b_i \otimes y_i \right\|_{S_p(X_{C_p})} &\geq \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j \left(\sum_{i=1}^{f(k)} b_i \otimes y_i \right) \right\|_{S_p(C_p(X_n))} \\
&= \theta \left\| \sum_{j=1}^{f(k)} b_j \otimes e_{j1} \otimes y_j \right\|_{S_p(C_p(X_n))} \\
&\geq \theta \left\| \sum_{j=1}^{f(k)} b_j \otimes e_{j1} \otimes t_{n_j} \right\|_{S_p(C_p(X_n))} \\
&\geq \theta \left\| \sum_{j=1}^{f(k)} b_j \otimes e_{j1} \right\|_{S_p(C_p)}
\end{aligned}$$

by Lemma 3.6 and 3.7 below for any $(b_i)_{i=1}^{f(k)} \subseteq S_p$.

For the right inequality we will show the following. For any disjoint and normalized sequence $(y_j)_{j=1}^{f(k)} \subseteq X_{C_p}$ we have

$$(3.1) \quad \left\| \sum_{j=1}^{f(k)} b_j \otimes y_j \right\|_{S_p(X_n)} \leq \left\| \sum_{j=1}^{f(k)} b_j \otimes e_{j1} \right\|_{S_p(C_p)}$$

for all $(b_j)_{j=1}^{f(k)} \subseteq S_p$.

Let us use induction on n . When $n = 0$ we are done since

$$CB(C_p, R_p +_p C_p) = B(C_p, R_p +_p C_p)$$

isometrically and $(e_{j1})_{j \geq 1}$ and $(y_j)_{j=1}^{f(k)}$ are orthonormal. Suppose we have (3.1) for n . Then for any “allowable” sequence $\{E_j\}_{j=1}^{f(l)} \subseteq \mathbb{N}$ we have

$$\begin{aligned} & \theta \left\| \sum_{j=1}^{f(l)} e_{j1} \otimes E_j \left(\sum_{i=1}^{f(k)} b_i \otimes y_i \right) \right\|_{S_p(C_p(X_n))} \\ &= \left\| \sum_{j=1}^{f(l)} \sum_{i=1}^{f(k)} b_i \otimes e_{j1} \otimes \theta E_j y_i \right\|_{S_p(C_p(X_n))} \\ &\leq \left\| \sum_{j=1}^{f(l)} \sum_{i=1}^{f(k)} b_i \otimes e_{j1} \otimes \|\theta E_j y_i\| e_{1,ij} \right\|_{S_p(C_p(C_p))} \\ &= \left\| \left(\sum_{j=1}^{f(l)} \sum_{i=1}^{f(k)} b_i^* b_i \|\theta E_j y_i\|^2 \right)^{\frac{1}{2}} \right\|_{S_p} \leq \left\| \left(\sum_{i=1}^{f(k)} b_i^* b_i \right)^{\frac{1}{2}} \right\|_{S_p}. \end{aligned}$$

The last inequality comes from

$$\sum_{j=1}^{f(l)} \|\theta E_j y_i\|^2 \leq \theta^2 \|y_i\|^2 \leq 1.$$

This conclude (3.1) for $n + 1$ as before. \square

Lemma 3.6. Let $(y_j)_{j=1}^{f(k)}$, $(E_j)_{j=1}^{f(k)}$ and $(n_j)_{j=1}^{f(k)}$ be the same as in Proposition 3.5. Then we have

$$(3.2) \quad \left\| \sum_{j=1}^{f(k)} b_j \otimes t_{n_j} \right\|_{S_p(X_n)} \leq \left\| \sum_{j=1}^{f(k)} b_j \otimes y_j \right\|_{S_p(C_p)}$$

for all $(b_j)_{j=1}^{f(k)} \subseteq S_p$.

Proof. We use induction on n . When $n = 0$ we are done since $R_p +_p C_p$ is a homogeneous Hilbertian operator space and $(t_{n_j})_{j=1}^{f(k)}$ and $(y_j)_{j=1}^{f(k)}$ are orthonormal. Suppose we have (3.2) for n , and consider any fixed “allowable” sequence $\{F_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$. If we set

$$x = \sum_{j=1}^{f(k)} b_j \otimes t_{n_j}, \quad y = \sum_{j=1}^{f(k)} b_j \otimes y_j$$

and

$$G_j = \{n_i : n_i \in F_j\} \text{ for } 1 \leq j \leq f(k),$$

then

$$G_j y = \sum_{n_i \in F_j} b_i \otimes t_{n_i}$$

and $\{G_j\}_{j=1}^{f(k)}$ is an “allowable” sequence. Consequently, we have

$$\begin{aligned} \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes F_j x \right\|_{S_p(C_p(X_n))} &= \theta \left\| \sum_{j=1}^{f(k)} \sum_{n_i \in F_j} b_i \otimes e_{j1} \otimes t_{n_i} \right\|_{S_p(C_p(X_n))} \\ &= \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes G_j y \right\|_{S_p(C_p(X_n))} \\ &\leq \|y\|_{S_p(X_{n+1})}. \end{aligned}$$

The last line is obtained by the complete contraction

$$X_{n+1} \rightarrow C_p(X_n), \quad z \mapsto \theta \sum_{j=1}^{f(k)} e_{j1} \otimes G_j z.$$

This conclude (3.2) for $n+1$ as before. \square

Lemma 3.7. *We have*

$$\left\| \sum_{j \geq 1} b_j \otimes e_{j1} \otimes t_j \right\|_{S_p(C_p(X_n))} \geq \left\| \sum_{j \geq 1} b_j \otimes e_{j1} \right\|_{S_p(C_p)}$$

for all finitely supported $(b_j)_{j \geq 1} \subseteq S_p$.

Proof. Consider a finitely supported $(b_j)_{j \geq 1} \subseteq S_p$. Then we have

$$\left\| \sum_{j \geq 1} b_j \otimes e_{j1} \otimes t_j \right\|_{S_p(C_p(X_n))} \geq \left\| \sum_{j \geq 1} b_j \otimes e_{j1} \otimes e_j \right\|_{S_p(C_p(R_p +_p C_p))}.$$

Note that $C_p(R_p +_p C_p) \cong C_p(R_p) +_p C_p(C_p) \cong S_p +_p C_p$ completely isometrically. Since the formal identities $\ell_1 \rightarrow C_1$ and $\ell_2 \rightarrow C_2$ are complete contractions so is $\ell_p \rightarrow C_p$ by complex interpolation. Since $\overline{\text{span}}\{e_{j1} \otimes e_j\}_{j \geq 1}$ correspond to ℓ_p and C_p in S_p and $C_p(C_p)$, respectively, we have

$$\overline{\text{span}}\{e_{j1} \otimes e_j\}_{j \geq 1} (\subseteq C_p(R_p +_p C_p)) \cong C_p$$

completely isometrically. Thus, we have

$$\left\| \sum_{j \geq 1} b_j \otimes e_{j1} \otimes t_j \right\|_{S_p(C_p(X_n))} = \left\| \sum_{j \geq 1} b_j \otimes e_{j1} \right\|_{S_p(C_p)}.$$

\square

3.2. X_{C_p} is a nontrivial weak- C_p space.

Proposition 3.8. *For $n \in \mathbb{N}$ we consider*

$$Y_n = \overline{\text{span}}\{t_i\}_{i \geq n+1} \subseteq X_{C_p}.$$

Then for any $E \subseteq Y_n$ with $\dim E = n$, we have

$$d_{cb}(E, C_p^n) \leq 3\theta^{-1}.$$

Proof. By Proposition V.6 of [1] there is a linear map $V : E \rightarrow Y_n$ such that $V(E) \subseteq \overline{\text{span}}\{y_i\}_{i=1}^{f(n)}$, where y_i 's are disjoint elements in Y_n and

$$\|Vf - f\| \leq \frac{1}{2n} \|f\|$$

for all $f \in E$. Now we consider the Auerbach basis $(x_i, x_i^*)_{i=1}^n$ of E . Then we have

$$\sum_{i=1}^{3n} \|x_i^*\| \|x_i - Vx_i\| \leq \frac{1}{2n} \sum_{i=1}^n \|x_i\| \leq \frac{1}{2},$$

which implies

$$d_{cb}(E, V(E)) \leq 3$$

by the perturbation lemma (Lemma 2.13.2 of [10]). On the other hand we have

$$d_{cb}(\overline{\text{span}}\{y_i\}_{i=1}^{f(n)}, C_p^{f(n)}) \leq \theta^{-1}$$

by Proposition 3.5. Consequently, by combining these we get our desired result. \square

Remark 3.9. The employment of Proposition V.6 of [1] in the proof of Proposition 3.8 is the reason why we have chosen $f(k) = (4k^3)^k$.

Actually we can show that every n -dimensional subspace of Y_n in Proposition 3.8 is completely complemented with bounded constants, so that we are ready to prove one of our main results.

Theorem 3.10. X_{C_p} is a weak- C_p space.

Proof. Let Y_n be the same as in Proposition 3.8. Consider any $E \subseteq X_{C_p}$ with

$$\dim E = 2n \text{ or } 2n + 1.$$

Then we have $\dim(E \cap Y_n) \geq n$, so that there is $F \subseteq E$ such that $\dim F = n$ and $F \subseteq Y_n$. Then by Proposition 3.8 we have

$$d_{F,cb}^H \leq 3\theta^{-1}.$$

Moreover, F is $3\theta^{-1}$ -completely complemented in Y_n by Proposition 2.3. Since Y_n itself is 1-completely complemented in X_{C_p} so is F , which implies X_{C_p} is a weak- C_p space. \square

All we have to do now is to show that X_{C_p} is not completely isomorphic to C_p . Before that we need to prepare the following lemmas which are analogues of Lemma 13.6 and 13.7 in [7].

Lemma 3.11. Let $N \in \mathbb{N}$ be fixed. Then, for any $n \geq 0$ and any $y, z \in C_p \otimes X_{C_p}$ with

$$\text{supp } y \subseteq \{1, 2, \dots, N\} \text{ and } \text{supp } z \subseteq \{N+1, N+2, \dots\}$$

we have

$$\|y + z\|_{C_p \otimes_{\min} X_{n+1}} \leq \max\{\|y\|_{C_p \otimes_{\min} X_{n+1}} + \alpha \|z\|_{C_p \otimes_{\min} X_n}, \|z\|_{C_p \otimes_{\min} X_{n+1}}\},$$

where $\alpha = \max\{1, \theta f(N)\}$.

Proof. Let us fix $n \geq 0$ and $y, z \in C_p \otimes X_{C_p}$ with

$$\text{supp } y \subseteq \{1, 2, \dots, N\} \text{ and } \text{supp } z \subseteq \{N+1, N+2, \dots\}.$$

Let T^y, T^z and T^{y+z} are linear maps from R_p into X_{C_p} associated with y, z and $y + z$, respectively.

Then we have

$$\|y + z\|_{C_p \otimes_{\min} X_{n+1}} = \|T^{y+z} : R_p \rightarrow X_{n+1}\|_{cb}.$$

For any $x = \sum_i x_i \otimes e_{1i} \in \mathcal{K} \otimes R_p$ we consider $\|T^{y+z}(x)\|_{n+1}$. If

$$\|T^{y+z}(x)\|_{n+1} = \|T^{y+z}(x)\|_n,$$

then we have

$$\begin{aligned}
\|T^{y+z}(x)\|_{n+1} &\leq \|T^y(x)\|_n + \|T^z(x)\|_n \\
&\leq \left(\|T^y : R_p \rightarrow X_n\|_{cb} + \|T^z : R_p \rightarrow X_n\|_{cb} \right) \|x\|_{\mathcal{K} \otimes R_p} \\
&\leq \left(\|y\|_{C_p \otimes \min X_{n+1}} + \|z\|_{C_p \otimes \min X_n} \right) \|x\|_{\mathcal{K} \otimes R_p}.
\end{aligned}$$

If not, we consider any “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$. When $k > N$, we have

$$\begin{aligned}
\theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j[T^{y+z}(x)] \right\|_{\mathcal{K} \otimes \min C_p(X_n)} &= \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j[T^z(x)] \right\|_{\mathcal{K} \otimes \min C_p(X_n)} \\
&\leq \|T^z(x)\|_{n+1} \leq \|z\|_{C_p \otimes \min X_{n+1}} \|x\|_{\mathcal{K} \otimes R_p}.
\end{aligned}$$

Otherwise, we have

$$\begin{aligned}
&\theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j[T^{y+z}(x)] \right\|_{\mathcal{K} \otimes \min C_p(X_n)} \\
&= \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j[T^y(x)] \right\|_{\mathcal{K} \otimes \min C_p(X_n)} + \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j[T^z(x)] \right\|_{\mathcal{K} \otimes \min C_p(X_n)} \\
&\leq \|T^y(x)\|_{n+1} + \theta \sum_{j=1}^{f(k)} \|E_j[T^z(x)]\|_n \\
&\leq \left(\|y\|_{C_p \otimes \min X_{n+1}} + \theta N \|z\|_{C_p \otimes \min X_n} \right) \|x\|_{\mathcal{K} \otimes R_p}.
\end{aligned}$$

Combining the above results we get our desired estimate. □

Lemma 3.12. *For any $n \geq 0$ and any $x \in \mathcal{K} \otimes_{\min} X_{C_p}$ we have*

$$(3.3) \quad \|x\|_{\mathcal{K} \otimes_{\min} X_{C_p}} \leq \|(x, \theta^n x)\|_{\mathcal{K} \otimes_{\min}(X_n \oplus_p C_p)}.$$

Proof. We will use induction on n to show (3.3) for all $x \in \mathcal{K}_0 \otimes c_{00}$.

When $n = 0$, it is trivial. Suppose we have (3.3) for all $x \in \mathcal{K}_0 \otimes c_{00}$ and for a fixed $n \geq 0$, which is equivalent to the fact that the map

$$(x, \theta^n x) \mapsto x, F \rightarrow X_{C_p}$$

is a complete contraction, where $F = \overline{\text{span}}\{(y, \theta^n y) : y \in c_{00}\} \subseteq X_n \oplus_p C_p$. Then, for any “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$ we have another complete contraction

$$e_{j1} \otimes (x, \theta^n x) \mapsto e_{j1} \otimes x, C_p(F) \rightarrow C_p(X_{C_p}).$$

Thus, we have

$$\begin{aligned}
& \theta \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x \right\|_{\mathcal{K} \otimes_{\min} C_p(X_{C_p})} \\
& \leq \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes (\theta E_j x, \theta^{n+1} E_j x) \right\|_{\mathcal{K} \otimes_{\min} C_p(X_n \oplus_p C_p)} \\
& = \left\| \left(\sum_{j=1}^{f(k)} e_{j1} \otimes \theta E_j x, \sum_{j=1}^{f(k)} e_{j1} \otimes \theta^{n+1} E_j x \right) \right\|_{\mathcal{K} \otimes_{\min} [C_p(X_n) \oplus_p C_p(C_p)]} \\
& \leq \|(x, \theta^{n+1} x)\|_{\mathcal{K} \otimes_{\min} (X_{n+1} \oplus_p C_p)}
\end{aligned}$$

The last line follows from the fact that the maps

$$x \mapsto \theta \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x, \quad X_{n+1} \rightarrow C_p(X_n)$$

and

$$x \mapsto \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x, \quad C_p \rightarrow C_p(C_p)$$

are complete contractions.

Since

$$\|x\|_{m+1} = \max \left\{ \|x\|_0, \theta \sup \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x \right\|_{\mathcal{K} \otimes_{\min} C_p(X_m)} \right\}$$

for all $m \geq 0$ by Proposition 3.3, we have

$$\|x\| \leq \max \left\{ \|x\|_0, \theta \sup \left\| \sum_{j=1}^{f(k)} e_{j1} \otimes E_j x \right\|_{\mathcal{K} \otimes_{\min} C_p(X_{C_p})} \right\},$$

which leads us to our desired conclusion. \square

The following proposition is the crucial point to explain why we cannot have complete isomorphism between C_p and X_{C_p} .

Proposition 3.13. *Let $n \geq 0$. Then, $C_p \otimes_{\min} E$ contains an isomorphic copy of c_0 for any infinite dimensional subspace $E \subseteq X_n$.*

Proof. We will use induction on n . Consider $n = 0$. Since

$$C_p \otimes_{\min} (R_p +_p C_p) \hookrightarrow B(\ell_2), \quad e_{j1} \otimes e_j \mapsto e_{jj}$$

$(e_{j1} \otimes e_j)_{j \geq 1}$ is a basic sequence in $C_p \otimes_{\min} (R_p +_p C_p)$ equivalent to the canonical basis of c_0 . Since $R_p +_p C_p$ is homogeneous, every infinite dimensional subspace $E \subseteq R_p +_p C_p$ is completely isometric to $R_p +_p C_p$ itself. Thus, we get the desired result for $n = 0$.

Now suppose that $C_p \otimes_{\min} E$ contains an isomorphic copy of c_0 for any infinite dimensional subspace $E \subseteq X_n$. Let $F \subseteq X_{n+1}$ be infinite dimensional, and $\epsilon > 0$ be arbitrarily given. We claim that there is an infinite dimensional subspace $F' \subseteq F$ such that $C_p \otimes_{\min} F'$ and $C_p \otimes_{\min} X_n|_{F'}$ are isomorphic or we can choose a sequence $(x_i)_{i \geq 1} \subseteq C_p \otimes_{\min} F$ satisfying

$$\|x_i\|_{C_p \otimes_{\min} X_{n+1}} = 1$$

for all $i \geq 1$ and

$$\left\| \sum_{i=1}^M x_i \right\|_{C_p \otimes_{\min} X_{n+1}} \leq 1 + \epsilon$$

for all $M \geq 1$. Both cases imply $C_p \otimes_{\min} F$ contains an isomorphic copy of c_0 , and consequently we get our desired induction result.

For the claim we start with a norm 1 vector $x_1 = e_{11} \otimes x \in C_p \otimes_{\min} F$. Suppose we have disjoint and finitely supported $x_1, \dots, x_m \in C_p \otimes_{\min} F$ with

$$\|x_i\|_{C_p \otimes_{\min} X_{n+1}} = 1$$

for all $1 \leq i \leq m$ and

$$\left\| \sum_{i=1}^m x_i \right\|_{C_p \otimes_{\min} X_{n+1}} \leq 1 + \epsilon \sum_{i=1}^{m-1} \frac{1}{2^i}.$$

Let N be a natural number such that $N \geq \text{supp} x_i$ for all $1 \leq i \leq m$ and

$$Y_N = \overline{\text{span}}\{e_i\}_{i \geq N+1} \subseteq X_{n+1}.$$

If $C_p \otimes_{\min} F \cap Y_N$ and $C_p \otimes_{\min} X_n|_{F \cap Y_N}$ are isomorphic, then it is done by the induction hypothesis. Suppose $C_p \otimes_{\min} F \cap Y_N$ and $C_p \otimes_{\min} X_n|_{F \cap Y_N}$ are not isomorphic. Then there is a finitely supported

$$x_{m+1} \in C_p \otimes_{\min} F \cap Y_N \text{ with } \text{supp} x_{m+1} \subseteq \{N+1, N+2, \dots\}$$

satisfying

$$\|x_{m+1}\|_{C_p \otimes_{\min} X_{n+1}} = 1 \text{ and } \|x_{m+1}\|_{C_p \otimes_{\min} X_n} < \frac{\epsilon}{2^m \alpha},$$

where $\alpha = \max\{1, \theta f(N)\}$. Then by Lemma 3.11 we have

$$\left\| \sum_{i=1}^{m+1} x_i \right\|_{C_p \otimes_{\min} X_{n+1}} \leq 1 + \epsilon \sum_{i=1}^m \frac{1}{2^i}.$$

By repeating this process we get our claim. \square

Theorem 3.14. X_{C_p} is not completely isomorphic to C_p .

Proof. Suppose that X_{C_p} is C -completely isomorphic to C_p for some $C > 0$. Then X_{C_p} is $(C + \epsilon)$ -homogeneous for any $\epsilon > 0$. Thus, by repeating the proof of Proposition 10.1 in [10] to X_{C_p} and C_p we get

$$\|I\|_{cb} = \|I\|_{cb} \|I^{-1}\|_{cb} \leq (C + \epsilon)^2 d_{cb}(X_{C_p}, C_p) \leq (C + \epsilon)^3,$$

where $I : X_{C_p} \rightarrow C_p$ is the formal identity. By Lemma 3.12 and (2.1) we have

$$\begin{aligned} (C + \epsilon)^{-3p} \|x\|_{S_p(C_p)}^p &\leq \|x\|_{S_p(X_{C_p})}^p \leq \|(x, \theta^n x)\|_{S_p(X_n \oplus_p C_p)}^p \\ &= \|x\|_{S_p(X_n)}^p + \theta^{pn} \|x\|_{S_p(C_p)}^p \end{aligned}$$

for any $x \in \mathcal{K}_0 \otimes c_{00}$.

If we choose n large enough so that $\theta^{pn} < \frac{(C+\epsilon)^{-3p}}{2}$, then we get

$$\frac{(C + \epsilon)^{-3p}}{2} \|x\|_{S_p(C_p)}^p \leq \|x\|_{S_p(X_n)}^p.$$

Consequently, we have

$$d_{cb}(X_n, C_p) \leq \frac{(C + \epsilon)^3}{2^{\frac{1}{p}}}.$$

However, since $C_p \otimes_{\min} C_p \subseteq CB(R_p, C_p) \cong S_{\frac{2p}{2-p}}$ isometrically and $S_{\frac{2p}{2-p}}$ is a reflexive Banach space with a basis $C_p \otimes_{\min} X_n$ does not contain any isomorphic copy of c_0 , which is contradictory to Proposition 3.13. \square

Remark 3.15. Note that if a weak- H space is a homogeneous Hilbertian space, then it is completely isomorphic to H itself by Proposition 3.8. of [4]. Thus, X_{C_p} is not homogeneous.

4. THE CASE $p = 2$

4.1. The construction and basic properties of the canonical basis. We will construct X_{OH} , an example of nontrivial weak- OH space, in a similar way. Many arguments used in section 3 still work for OH case also, so that we only provide the proofs which we need to approach in a significantly different way.

Consider a fixed constant $0 < \theta < 1$. For $x \in \mathcal{K}_0 \otimes c_{00}$ we define

$$\|x\|_0 = \|x\|_{\mathcal{K} \otimes_{\min} (\min \ell_2)}$$

and for $n \geq 0$

$$\|x\|_{n+1} = \max \left\{ \|x\|_n, \theta \sup \left\| (E_j x)_{j=1}^{f(k)} \right\|_{\mathcal{K} \otimes_{\min} \ell_2^{f(k)}(X_n)} \right\},$$

where the inner supremum runs over all “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$. As before we denote the completion of $(c_{00}, \|\cdot\|_n)$ by X_n and X_{n+1} inherits the operator space structure from $X_n \oplus_\infty \ell_\infty(I; \{\ell_2^{f(k)}(X_n)\})$, where I is the collection of all allowable sequences.

Proposition 4.1. *For any $x \in \mathcal{K}_0 \otimes c_{00}$, $(\|x\|_n)_{n \geq 0}$ is increasing, and we have*

$$\|x\|_{\mathcal{K} \otimes_{\min} (\min \ell_2)} \leq \|x\|_n \leq \|x\|_{\mathcal{K} \otimes_{\min} OH}$$

for all $n \geq 0$.

Now we can consider $\|x\| = \lim_{n \rightarrow \infty} \|x\|_n$ for all $x \in \mathcal{K}_0 \otimes c_{00}$, and X_{OH} , the completion of $(c_{00}, \|\cdot\|)$ inherits the operator space structure from $\ell_\infty(X_n)$. Moreover, X_{OH} is Hilbertian by Proposition 4.1.

Proposition 4.2. *For any $x \in \mathcal{K}_0 \otimes c_{00}$ and any $n \geq 0$ we have*

$$\|x\|_{n+1} = \max \left\{ \|x\|_0, \theta \sup \left\| (E_j x)_{j=1}^{f(k)} \right\|_{\mathcal{K} \otimes_{\min} \ell_2^{f(k)}(X_n)} \right\},$$

where the inner supremum runs over all “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$.

Proposition 4.3. *The canonical basis $\{t_i\}_{i \geq 1}$ is a normalized 1-completely unconditional basis for X_{OH} .*

Proof. The only different point from the proof of Proposition 3.4 is the proof for X_0 . Indeed, for any $\sum_{i \geq 1} x_i \otimes t_i \in \mathcal{K}_0 \otimes c_{00}$ we have

$$\begin{aligned} \left\| \sum_{i \geq 1} a_i x_i \otimes t_i \right\|_0 &= \left\| \sum_{i \geq 1} a_i x_i \otimes e_i \right\|_{\mathcal{K} \otimes_{\min} \min \ell_2} = \|u : \ell_2 \rightarrow \mathcal{K}, e_i \mapsto a_i x_i\| \\ &= \sup \left\{ \left\| \sum_{i \geq 1} \xi_i a_i x_i \right\| : \sum_{i \geq 1} |\xi_i|^2 \leq 1 \right\} \\ &\leq \|v : \ell_2 \rightarrow \mathcal{K}, e_i \mapsto x_i\| \leq \left\| \sum_{i \geq 1} x_i \otimes t_i \right\|_0 \end{aligned}$$

since $\sum_{i \geq 1} |\xi_i a_i|^2 \leq 1$. □

Now we investigate operator space structure spanned by certain disjoint block sequences of $\{t_i\}_{i \geq 1}$. They are θ -completely isomorphic to operator Hilbert spaces with the same dimensions.

Proposition 4.4. *Let $(y_j)_{j=1}^{f(k)}$ be a disjoint and normalized block sequence of $\{t_i\}_{i \geq 1}$ such that $\text{supp}(y_j) \subseteq \{k, k+1, \dots\}$ for all $1 \leq j \leq f(k)$. Then we have*

$$\theta \left(\sum_{j=1}^{f(k)} \|b_j\|_{S_2}^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{j=1}^{f(k)} b_j \otimes y_j \right\|_{S_2(X_{OH})} \leq \left(\sum_{j=1}^{f(k)} \|b_j\|_{S_2}^2 \right)^{\frac{1}{2}}.$$

Proof. Consider any “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$. Since we have

$$X_{n+1} \rightarrow \ell_2^{f(k)}(X_n), \quad x \mapsto (\theta E_j x)_{j=1}^{f(k)}$$

is completely contractive

$$X_{OH} \rightarrow \ell_2^{f(k)}(X_n), \quad x \mapsto (\theta E_j x)_{j=1}^{f(k)}$$

is also completely contractive. Thus if we set $E_j = \text{supp}(y_j)$, then $\{E_j\}_{j=1}^{f(k)}$ is “allowable”, so that we have

$$\begin{aligned} \left\| \sum_{j=1}^{f(k)} b_j \otimes y_j \right\|_{S_2(X_{OH})} &\geq \left\| (\theta E_j \left(\sum_{i=1}^{f(k)} b_i \otimes y_i \right))_{j=1}^{f(k)} \right\|_{S_2(\ell_2^{f(k)}(X_n))} \\ &= \theta \left(\sum_{j=1}^{f(k)} \left\| E_j \left(\sum_{i=1}^{f(k)} b_i \otimes y_i \right) \right\|_{S_2(X_n)}^2 \right)^{\frac{1}{2}} \\ &= \theta \left(\sum_{j=1}^{f(k)} \|b_j \otimes y_j\|_{S_2(X_n)}^2 \right)^{\frac{1}{2}} \\ &= \theta \left(\sum_{j=1}^{f(k)} \|b_j\|_{S_2}^2 \|y_j\|_{X_n}^2 \right)^{\frac{1}{2}} = \theta \left(\sum_{j=1}^{f(k)} \|b_j\|_{S_2}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The first and the third equality comes from Proposition 2.1 and Lemma 3.6 of [9], respectively.

For the right inequality we will show the following more general results using induction on n .

$$(4.1) \quad \left\| \sum_{i=1}^{f(k)} b_i \otimes y_i \right\|_{S_2(X_n)} \leq \left(\sum_{i=1}^{f(k)} \|b_i\|_{S_2}^2 \right)^{\frac{1}{2}}$$

for all $(b_i)_{i=1}^{f(k)} \subseteq \mathcal{K}$ and for any disjoint and normalized sequence $(y_i)_{i=1}^{f(k)}$.

When $n = 0$, we have (4.1) since

$$CB(OH, \min \ell_2) = B(OH, \min \ell_2)$$

isometrically and $(e_{j1})_{j \geq 1}$ and $(y_j)_{j=1}^{f(k)}$ are orthonormal.

Now suppose we have (4.1) for n . By the induction hypothesis and Proposition 2.1 of [9] we have for any “allowable” sequence $\{E_j\}_{j=1}^{f(l)} \subseteq \mathbb{N}$ that

$$\begin{aligned} \theta \left\| (E_j(\sum_{i=1}^{f(k)} b_i \otimes y_i))_{j=1}^{f(l)} \right\|_{S_2(\ell_2^{f(l)}(X_n))} &= \left(\sum_{j=1}^{f(l)} \left\| \sum_{i=1}^{f(k)} b_i \otimes \theta E_j y_i \right\|_{S_2(X_n)}^2 \right)^{\frac{1}{2}} \\ &\leq \left[\sum_{j=1}^{f(l)} \left(\sum_{i=1}^{f(k)} \|b_i\|_{S_2}^2 \|\theta E_j y_i\|_{X_n}^2 \right) \right]^{\frac{1}{2}} \\ &= \left[\sum_{i=1}^{f(k)} \|b_i\|_{S_2}^2 \left(\sum_{j=1}^{f(l)} \|\theta E_j y_i\|_{X_n}^2 \right) \right]^{\frac{1}{2}} \\ &\leq \left[\sum_{i=1}^{f(k)} \|b_i\|_{S_2}^2 \|y_i\|_{X_{OH}}^2 \right]^{\frac{1}{2}} = \left[\sum_{i=1}^{f(k)} \|b_i\|_{S_2}^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Thus, we have that

$$e_i \mapsto (E_j y_i)_{j=1}^{f(l)}, OH_{f(k)} \rightarrow \ell_2^{f(l)}(X_n)$$

is a complete contraction for all “allowable” sequence $\{E_j\}_{j=1}^{f(l)}$, which implies

$$e_i \mapsto y_i, OH_{f(k)} \rightarrow X_{n+1}$$

is also a complete contraction. Consequently, we get the desired induction result for $n+1$. \square

4.2. X_{OH} is a nontrivial weak- OH space.

Proposition 4.5. *For $n \in \mathbb{N}$ we consider*

$$Y_n = \overline{\text{span}}\{t_i\}_{i \geq n+1} \subseteq X_{OH}.$$

Then for any $E \subseteq Y_n$ with $\dim E = n$, we have

$$d_{cb}(E, OH_n) \leq 3\theta^{-1}.$$

Theorem 4.6. *X_{OH} is a weak- OH space.*

Lemma 4.7. *Let $N \in \mathbb{N}$ be fixed. Then, for any $n \geq 0$ and any $y, z \in X_{OH}$ with $\text{supp } y \subseteq \{1, 2, \dots, N\}$ and $\text{supp } z \subseteq \{N+1, N+2, \dots\}$ we have*

$$\|y + z\|_{n+1} \leq \max\{\|y\|_{n+1} + \alpha \|z\|_n, \|z\|_{n+1}\},$$

where $\alpha = \max\{1, \theta \sqrt{f(N)}\}$.

Proof. Let's fix $n \geq 0$. If $\|y + z\|_{n+1} = \|y + z\|_n$, then it is trivial.

Now we consider any “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$. When $k > N$, we have

$$E_j(y + z) = E_j z,$$

so that

$$\theta \left\| (E_j(y + z))_{j=1}^{f(k)} \right\|_{\mathcal{K} \otimes \min \ell_2^{f(k)}(X_n)} \leq \|z\|_{n+1}.$$

When $k \leq N$, we have

$$\begin{aligned}
& \theta \left\| (E_j(y+z))_{j=1}^{f(k)} \right\|_{\mathcal{K} \otimes_{\min} \ell_2^{f(k)}(X_n)} \\
& \leq \theta \left\| (E_j y)_{j=1}^{f(k)} \right\|_{\mathcal{K} \otimes_{\min} \ell_2^{f(k)}(X_n)} + \theta \left\| (E_j z)_{j=1}^{f(k)} \right\|_{\mathcal{K} \otimes_{\min} \ell_2^{f(k)}(X_n)} \\
& \leq \|y\|_{n+1} + \theta \left\| (E_j z)_{j=1}^{f(k)} \right\|_{\ell_2^{f(k)}(\mathcal{K} \otimes_{\min} X_n)} \\
& = \|y\|_{n+1} + \theta \sqrt{\sum_{j=1}^{f(k)} \|E_j z\|_n^2} \\
& \leq \|y\|_{n+1} + \theta \sqrt{f(N)} \|z\|_n.
\end{aligned}$$

□

Lemma 4.8. *For any $n \geq 0$ and any $x \in \mathcal{K} \otimes_{\min} X_{OH}$ we have*

$$\|x\|_{\mathcal{K} \otimes_{\min} X_{OH}} \leq \|(x, \theta^n x)\|_{\mathcal{K} \otimes_{\min} (X_n \oplus_2 OH)}.$$

Proposition 4.9. *For any $n \geq 0$ and for any infinite dimensional subspace $E \subseteq X_n$ we have that $R \otimes_{\min} E$ contains an isomorphic copy of c_0 .*

Theorem 4.10. *X_{OH} is not completely isomorphic to OH .*

Proof. The only different point from the proof of Theorem 3.14 is to observe that $R \otimes_{\min} OH$ does not contain any isomorphic copy of c_0 . Indeed, we have

$$\begin{aligned}
R \otimes_{\min} OH &= OH \otimes_h R = [R, C]_{\frac{1}{2}} \otimes_h R \\
&= [R \otimes_h R, C \otimes_h R]_{\frac{1}{2}}.
\end{aligned}$$

Since $R \otimes_h R$ and $C \otimes_h R$ are isometric to S_2 and S_∞ , respectively, $R \otimes_{\min} OH$ is isometric to S_4 . Thus, $R \otimes_{\min} OH$ is a reflexive Banach space with a basis, so that it does not contain a isomorphic copy of c_0 . □

Remark 4.11. Instead of $X_0 = \min \ell_2$ we can use $R + C$ in the above construction for another nontrivial Hilbertian weak- OH space Y_{OH} . However, we do not know X_{OH} and Y_{OH} are completely isomorphic or not at the time of this writing.

Remark 4.12. By a similar procedure we can construct a non-Hilbertian weak- H space T_{OH} . The construction is as follows.

We will define a sequence of norms on $\mathcal{K}_0 \otimes c_{00}$ again. Consider a fixed constant $0 < \theta < 1$. For $x \in \mathcal{K}_0 \otimes c_{00}$ we define

$$\|x\|'_0 = \|x\|_{\mathcal{K} \otimes_{\min} c_0}$$

and for $n \geq 1$ we define

$$\|x\|'_{n+1} = \max \left\{ \|x\|'_n, \theta \sup \left\| (E_j x)_{j=1}^{f(k)} \right\|_{\mathcal{K} \otimes_{\min} \ell_2^{f(k)}(T_n)} \right\},$$

where the inner supremum runs over all “allowable” sequence $\{E_j\}_{j=1}^{f(k)} \subseteq \mathbb{N}$. We denote the completion of $(c_{00}, \|\cdot\|'_n)$ by T_{n+1} , and T_{n+1} inherits the operator space structure from $T_n \oplus_\infty \ell_\infty(I; \{\ell_2^{f(k)}(T_n)\})$, where I is the collection of all allowable sequences.

Then by a similar argument as in Proposition 4.1 we can show that

$$\|x\|_{\mathcal{K} \otimes_{\min} c_0} \leq \|x\|'_n \leq \|x\|_{\mathcal{K} \otimes_{\min} OH}$$

for all $n \geq 0$ and $(\|x\|'_n)_{n \geq 0}$ is increasing. Thus,

$$\|x\|' = \lim_{n \rightarrow \infty} \|x\|'_n$$

converges for all $x \in \mathcal{K}_0 \otimes c_{00}$, so that T_{OH} , the completion of $(c_{00}, \|\cdot\|')$ inherits the operator space structure from $\ell_\infty(T_n)$.

If we look at the underlying Banach space of T_{OH} , then it is nothing but a variant of modified 2-convexification of Tsirelson's space in (2) of Notes and Remarks in X.e (p.117 of [1]). The only difference is the fact that we replaced $f(k) = k$ into $f(k) = (4k^3)^k$, and it is well-known that our T_{OH} is isomorphic to (as a Banach space) the 2-convexified Tsirelson's space (see X.e. and Appendix b. in [1]), which is not isomorphic to any Hilbert space.

By a similar argument we can show T_{OH} is a non-Hilbertian example of weak- OH space.

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